Determinantal representations of invariant hyperbolic plane curves

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Background

- A *determinantal representation* of $f \in \mathbb{R}[t,x,y]_d$ is a $d \times d$ matrix $M = tM_0 + xM_1 + yM_2$ such that $f = \det(M)$. It is called *definite* if there exists $e \in \mathbb{R}^3$ so that $M(e) \succ 0$. Polynomials which have definite determinantal representations are called *hyperbolic*.

- A smooth projective curve of degree d is called *hyperbolic* if the real points in its zero set consist of a maximal number of nested ovals.

- The *cyclic group* of order n is $C_n = \langle \Phi \rangle$ and the *dihedral group* of order n is $D_n = \langle \Phi, \Gamma \rangle$ where

$$\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi/n) & \sin(2\pi/n) \\ 0 & -\sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix} \text{ and } \Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

- The groups C_n and D_n act on $\mathbb{R}[t,x,y]$ where Φ is a rotation around $[1:0:0]$ and Γ is a reflection across the line $y = 0$.

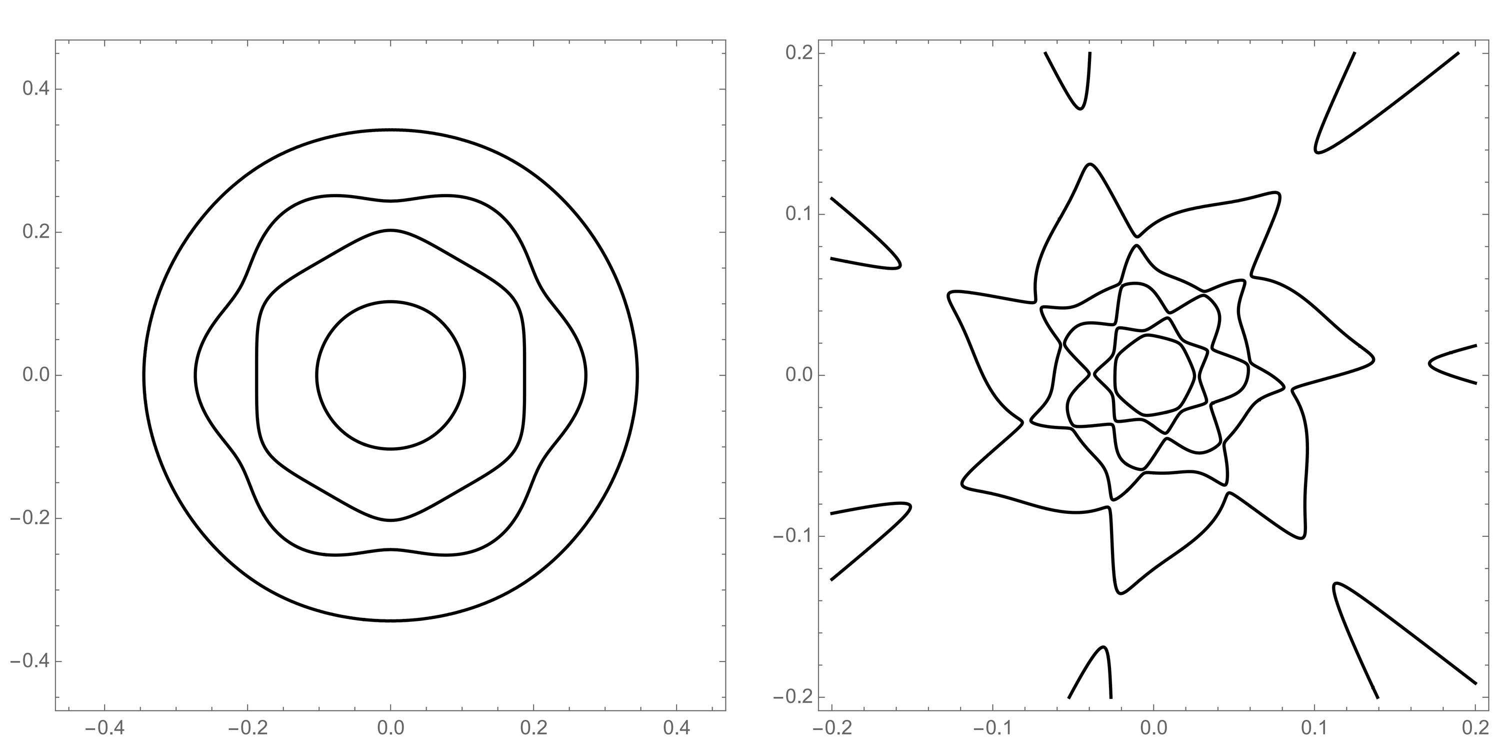

Figure 1: A hyperbolic curve of degree 8 invariant under D_6 in the hyperplane $\{t = 1\}$ (left) and a hyperbolic curve of degree 11 invariant under C_7 in the hyperplane $\{t = 1\}$ (right).

Main Problem and Results

Theorem ([1, 3]). If $A \in \mathbb{C}^{d\times d}$ satisfies

$$A_{ij} = 0 \text{ if } i - j \mod n \not\equiv -1, \tag{1}$$

then the plane curve defined by

$$f_A = \det\left(tI + x\left(\frac{A + A^*}{2}\right) + y\left(\frac{A - A^*}{2i}\right)\right) \tag{2}$$

is hyperbolic with respect to $(1,0,0)$ and invariant under the cyclic group of order n.

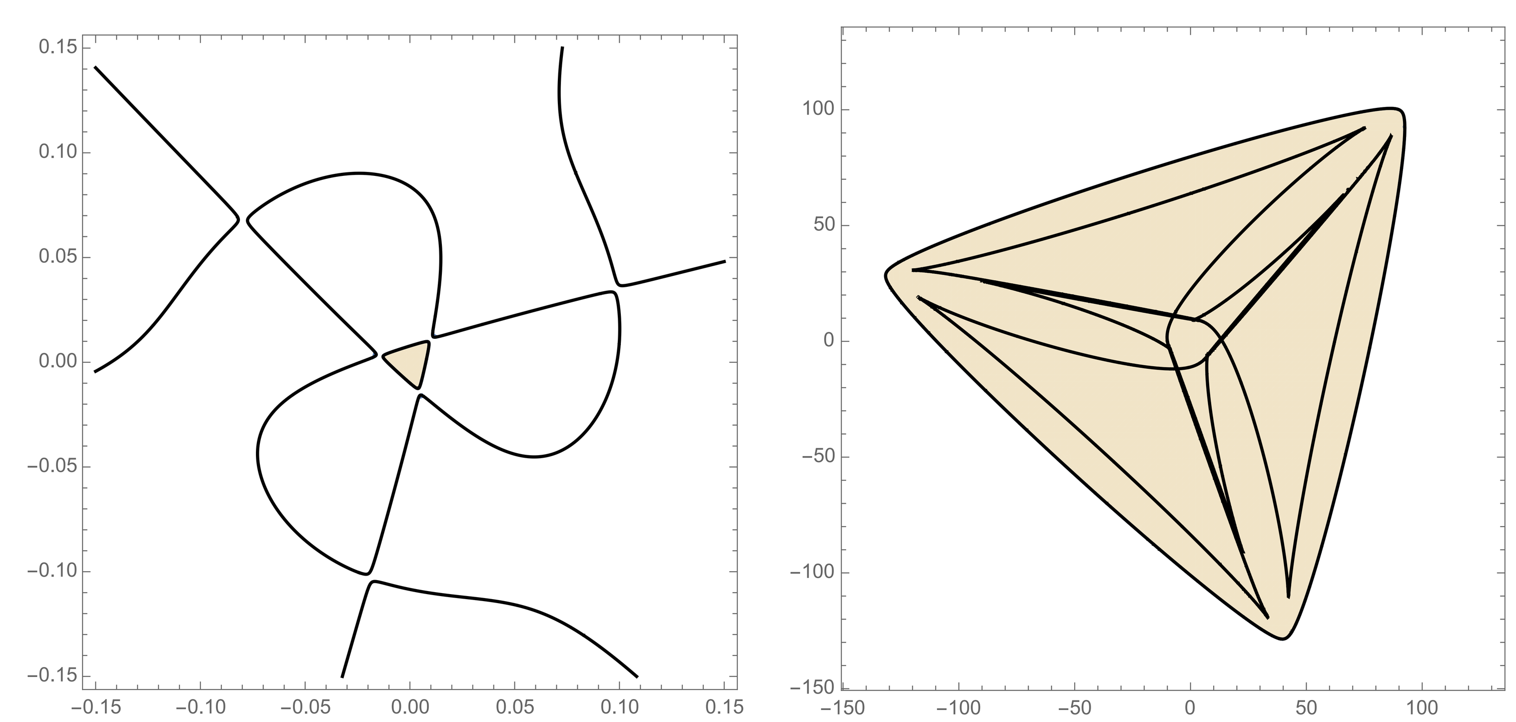

Figure 2: A hyperbolic quintic f_A invariant under C_3 in the hyperplane $\{t = 1\}$ (left) and the dual of f_A with shaded convex hull corresponding to $\mathcal{W}(A)$ (right).

Question [2]. If $f \in \mathbb{R}[t,x,y]_d$ is hyperbolic and invariant under C_n, does f have a determinantal representation of the form (2) where A satisfies (1)?

- We give a positive result in the case where $d = n$.

> **Theorem 1** (Lentzos–P [3]). Let $f \in \mathbb{R}[t,x,y]_n$ be monic and hyperbolic with respect to $(1,0,0)$. If f is invariant under C_n, then there exists $A \in \mathbb{C}^{n\times n}$ satisfying (1) such that $f = f_A$. Additionally, if f is invariant under D_n, then there exists $B \in \mathbb{R}^{n\times n}$ satisfying (1) such that $f = f_B$.

Problem with Generalization

- The hope is to generalize Theorem 1 for any $d > n$, but the construction only works for smooth curves.

- **Issue:** Invariant curves with $d \bmod n \geq 3$ always have multiple complex singularities, so "most" of these curves are singular!

Connection to the Numerical Range

- The *numerical range* of $A \in \mathbb{C}^{d\times d}$ is

$$\mathcal{W}(A) = \{x^*Ax \mid x \in \mathbb{C}^d, x^*x = 1\}. \tag{3}$$

- As a subset of $\mathbb{C} \cong \mathbb{R}^2$, $\mathcal{W}(A)$ is the convex hull of $g(1,x,y)$ where $\{g = 0\}$ is dual to the curve defined by $\{f_A = 0\}$ (see Figure 2).

> **Theorem 2** (Lentzos–P [3]). If $\mathcal{W}(A)$ is invariant under rotation by the angle $2\pi/n$ for any $A \in \mathbb{C}^{n\times n}$, then there exists $B \in \mathbb{C}^{n\times n}$ satisfying (1) such that $\mathcal{W}(B) = \mathcal{W}(A)$.

References

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